Q1

Find an expression for $\frac{dy}{dx}$ in terms of x and y for the following

(i) $e^{xy} + \ln(xy) = \operatorname{cosec}(x) + 4$

(ii) $4\cos(x^2y) - 3e^{x^2y} = 4e^y$

[5]

(alternatively, rewrite $\ln(xy)$ as $\ln x + \ln y$ before differenciating)

i) $\frac{d}{dx}$

$$\frac{d(xy)}{dx} e^{xy} + \frac{d(xy)}{dx}\frac{1}{xy} = -\operatorname{cosec}x\cot x + 0$$

$$\left(x\frac{dy}{dx} + y\right)e^{xy} + \left(x\frac{dy}{dx}+y\right)\frac{1}{xy} = -\operatorname{cosec}x\cot x$$

$$\frac{dy}{dx}\left(xe^{xy} + \frac{1}{y}\right) = -\operatorname{cosec}x\cot x - ye^{xy} - \frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{-\operatorname{cosec}x\cot x - ye^{xy} - \frac{1}{x}}{xe^{xy} + \frac{1}{y}}}$$

ii) $\frac{d}{dx}$

$$\frac{d(x^2y)}{dx}(-4\sin(x^2y)) + \frac{d(x^2y)}{dx}(-3e^{x^2y}) = 4e^y\frac{dy}{dx}$$

$$\left(x^2\frac{dy}{dx} + y2x\right)\left(-4\sin(x^2y) - 3e^{x^2y}\right) = 4e^y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(x^2(-4\sin(x^2y) - 3e^{x^2y}) - 4e^y\right)$$

$$= 2xy\left(4\sin(x^2y) + 3e^{x^2y}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{8xy\sin(x^2y) + 6xye^{x^2y}}{-4x^2\sin(x^2y) - 3x^2e^{x^2y} - 4e^y}}$$

Q2

[5]

$$\frac{d}{dx} \quad x^2 2y\frac{dy}{dx} + y^2 2x - 5 = 22\frac{dy}{dx} \quad ①$$

When $x = -2$

$$(-2)^2y^2 - 5(-2) = 22y$$

$$4y^2 - 22y + 10 = 0$$

$$2y^2 - 11y + 5 = 0$$

$$(2y-1)(y-5) = 0$$

$$y = \frac{1}{2}, 5$$

Since y must be an integer, $y = 5$.

Sub $x = -2$, $y = 5$ into ①

$$(-2)^2 2(5)\frac{dy}{dx} + 5^2 \times 2 \times -2 - 5 = 22\frac{dy}{dx}$$

$$40\frac{dy}{dx} - 100 - 5 = 22\frac{dy}{dx}$$

$$18\frac{dy}{dx} = 105$$

$$\frac{dy}{dx} = \frac{105}{18} = \boxed{\frac{35}{6}}$$

Q3

$$\frac{d}{dx} \qquad \frac{2x}{4} + \frac{1}{9}2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{2x}{4}}{\frac{2y}{9}} = -\frac{\cancel{2}x}{4} \times \frac{9}{\cancel{2}y} = \frac{-9x}{4y}$$

Sub in $y = kx$ to find the gradient where the ellipse meets the line:

$$\frac{dy}{dx} = \frac{-9\cancel{x}}{4k\cancel{x}} = \boxed{\frac{-9}{4k}}$$

The espression for $\frac{dy}{dx}$ does not involve (so is independant of) the values of x and y in the instances where the ellipse meets a line of the form $y = kx$ $(k \neq 0)$.

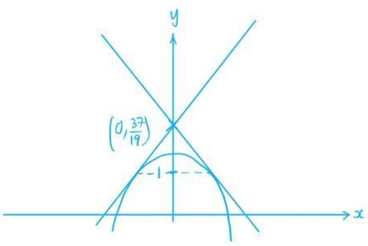
Q4

The curve C is described by the equation

$$\ln y + x^2y^2 = 9.$$

Show that the tangents of the two points on C where $y = 1$ meet at the point $\left(0, \frac{37}{19}\right)$.

[5]



$$y - y_1 = m(x - x_1)$$

When $y = 1$

$\ln 1 + x^2(1)^2 = 9$

$x^2 = 9$

$x = \pm 3$

$\therefore$ The two points are $(3, 1)$ and $(-3, 1)$

$$\frac{d}{dx} \quad \frac{1}{y}\frac{dy}{dx} + x^2 2y\frac{dy}{dx} + y^2 2x = 0$$

$$\frac{dy}{dx}\left(\frac{1}{y} + 2x^2y\right) = -2xy^2$$

$$\frac{dy}{dx} = \frac{-2xy^2}{\frac{1}{y} + 2x^2y}$$

At $(3, 1)$:

$$\frac{dy}{dx} = \frac{-2(3)1^2}{\frac{1}{1} + 2(3)^2(1)} = \frac{-6}{19}$$

$$y - 1 = -\frac{6}{19}(x - 3)$$

$$y = -\frac{6x}{19} + \frac{37}{19}$$

At $(-3, 1)$

$$\frac{dy}{dx} = \frac{-2(-3)(1)^2}{\frac{1}{1} + 2(-3)^2(1)} = \frac{6}{19}$$

$$y - 1 = \frac{6}{19}(x - (-3))$$

$$y = \frac{6}{19}x + \frac{37}{19}$$

The y-intercept of both tangents is $\frac{37}{19}$, therefore the two tangents meet at the point $(0, \frac{37}{19})$.

Q5a

a) For the two normals to be parallel, their gradients must be equal.

$$\text{gradient}_{(\text{at } x=-4)} = \text{gradient}_{(\text{at } x=4)}$$

$$\frac{d}{dx} \quad 6x + 2x\,3y^2\frac{dy}{dx} + y^3 2 + 0 = 0$$

$$\frac{dy}{dx}(6xy^2) = -6x - 2y^3$$

$$\frac{dy}{dx} = \frac{-6x-2y^3}{6xy^2} = \frac{-3x-y^3}{3xy^2}$$

when $x = 4$

$$3(4)^2 + 2(4)y^3 + 16 = 0$$
$$y^3 = -8$$
$$y = -2$$

$$\frac{dy}{dx} = \frac{-3(4)-(-2)^3}{3(4)(-2)^2} = \boxed{\frac{-1}{12}}$$

when $x = -4$

$$3(-4)^2 + 2(-4)y^3 + 16 = 0$$
$$y^3 = 8$$
$$y = 2$$

$$\frac{dy}{dy} = \frac{-3(-4)-8}{3(-4)(2)^2} = \boxed{\frac{-1}{12}}$$

We could use these tangent gradients to find the normal gradients, but it's easier to conclude...

> The gradients of the tangents are the same, so the tangents are parallel, and therefore the normals must be parallel.

Q5b

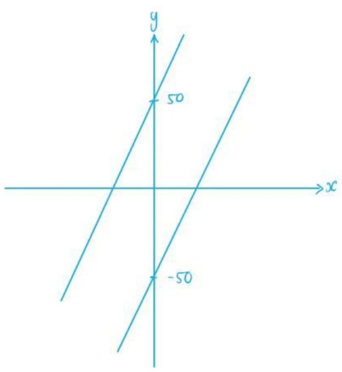
The curve C is described by the equation

$$3x^2 + 2xy^3 + 16 = 0.$$

(a) Show that the normal to C at the point where $x = -4$ is parallel to the normal to C at the point where $x = 4$. [4]

$\frac{dy}{dx} = \frac{-1}{12}$ where $(4, -2)$ and $(-4, 2)$

(b) Find the distance between the y-axis intercepts of these two normals.



b) $y - y_1 = m(x - x_1)$

$$m_n = \frac{-1}{\frac{dy}{dx}} = \frac{-1}{\frac{-1}{12}} = 12$$

Eqn of normal at $(4, -2)$:

$$y - (-2) = 12(x - 4)$$
$$y = 12x \boxed{-50}$$

Eqn of normal at $(-4, 2)$:

$$y - (2) = 12(x - (-4))$$
$$y = 12x \boxed{+50}$$

Find the difference between the two y-intercepts

$$\text{distance} = 50 - (-50) = \boxed{100}$$

Q6

Find the stationary points and determine their nature for the curve with equation $y^2 = 3x^2 - 2xy + 3$.

[8]

① $2y\frac{dy}{dx} = 6x - \left(2x\frac{dy}{dx} + y2\right)$

$\frac{dy}{dx}(2y + 2x) = 6x - 2y$

$\frac{dy}{dx} = \frac{6x-2y}{2y+2x} = \boxed{\frac{3x-y}{x+y}} = 0$ $\quad \frac{u}{v}$

② $3x - y = 0$

$y = 3x$

Sub $y = 3x$ into eqn for curve

$(3x)^2 = 3x^2 - 2x(3x) + 3$

$9x^2 = 3x^2 - 6x^2 + 3$

$12x^2 = 3$

$x^2 = \frac{1}{4}$ $\quad \therefore x = \sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

When $x = \frac{1}{2}$

$y = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

$\boxed{\left(\frac{1}{2}, \frac{3}{2}\right)}$

When $x = -\frac{1}{2}$

$y = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$

$\boxed{\left(-\frac{1}{2}, -\frac{3}{2}\right)}$

③ Classify stationary points

quotient rule

$\frac{d^2\left(\frac{u}{v}\right)}{dx^2} = \frac{(x+y)(3-\cancel{\frac{dy}{dx}}^{0}) - (3x-y)(1+\cancel{\frac{dy}{dx}}^{0})}{\boxed{(x+y)^2}} = \frac{4y}{(+)}$

Since this is a squared number the denominator will always be +ve!

At $\left(\frac{1}{2}, \frac{3}{2}\right)$ $\quad$ 2nd derivative: $\frac{4\left(\frac{3}{2}\right)}{(+)} > 0 \rightarrow$ $\boxed{\text{min point}}$

At $\left(-\frac{1}{2}, -\frac{3}{2}\right)$ $\quad$ 2nd derivative: $\frac{4\left(-\frac{3}{2}\right)}{(+)} < 0 \rightarrow$ $\boxed{\text{max point}}$

Q7

The curve C is defined by

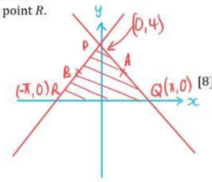
$$e^{\sin(xy)} = 1 \qquad \{y > 0\}$$

Points A and B have coordinates $\left(\frac{\pi}{2}, 2\right)$ and $\left(-\frac{\pi}{2}, 2\right)$ respectively.

The tangents to C at points A and B intersect at the point P.
The tangent to C at point A intersects the x-axis at point Q.
The tangent to C at point B intersects the x-axis at point R.

Find the area of triangle PQR.

[8]



$y - y_1 = m(x - x_1)$

$\frac{d}{dx}$ $\boxed{\frac{d(\sin(xy))}{dx}} e^{\sin(xy)} = 0$

$\boxed{\frac{d(xy)}{dx}} \cos(xy) e^{\sin(xy)} = 0$

$\left(x\frac{dy}{dx} + y\right)\cos(xy)e^{\sin(xy)} = 0$

$x\frac{dy}{dx}\cancel{\cos(xy)e^{\sin(xy)}} = -y\cancel{\left(\cos(xy)\right)e^{\sin(xy)}}$

$\frac{dy}{dx} = -\frac{y}{x}$

At A, $x = \frac{\pi}{2}$, $y = 2$

$\frac{dy}{dx} = -\frac{2}{\frac{\pi}{2}} = -\frac{4}{\pi}$

$y - 2 = -\frac{4}{\pi}\left(x - \frac{\pi}{2}\right)$

At B, $x = -\frac{\pi}{2}$, $y = 2$

$\frac{dy}{dx} = -\frac{2}{-\frac{\pi}{2}} = \frac{4}{\pi}$

$y - 2 = \frac{4}{\pi}\left(x + \frac{\pi}{2}\right)$

$-\cancel{\frac{4}{\pi}}\left(x - \frac{\pi}{2}\right) = \cancel{\frac{4}{\pi}}\left(x + \frac{\pi}{2}\right)$

$2x = 0$

$x = 0$

$y - 2 = -\frac{4}{\pi}\left(0 - \frac{\pi}{2}\right)$

$y - 2 = 2$ $\quad \therefore y = 4$ $\quad P(0, 4)$

At Q, $y = 0$

$0 - 2 = -\frac{4}{\pi}\left(x - \frac{\pi}{2}\right)$

$\frac{\pi}{2} = x - \frac{\pi}{2}$

$x = \pi$

$Q(\pi, 0)$

At R, $y = 0$

$0 - 2 = \frac{4}{\pi}\left(x + \frac{\pi}{2}\right)$

$-\frac{\pi}{2} = x + \frac{\pi}{2}$

$x = -\pi$

$R(-\pi, 0)$

Area $= \frac{1}{2}bh = \frac{1}{2}(2\pi)4 = \boxed{4\pi}$ units squared.

Q8

Let $y = a^{x^k}$

$\ln y = \ln a^{x^k}$

$\ln y = x^k \ln a$

$\frac{d}{dx} \quad \frac{1}{y}\frac{dy}{dx} = kx^{k-1}\ln a$

$\frac{dy}{dx} = ykx^{k-1}\ln a$

Sub $y = a^{x^k}$

$$\boxed{\frac{d\left(a^{x^k}\right)}{dx} = ka^{x^k}x^{k-1}\ln a}$$